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CONSTRUCTION AND CALCULATION OF A VARIABLE ACOUSTIC IMPEDANCE

BY

V. THORSEN

KØBENHAVN
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Introduction.

In the field of electricity, pure ohmic, inductive and capacitive resistances can rather easily be constructed; similar possibilities are, however, absent within acoustics. Here, the acoustic values corresponding to ohmic resistance, inductance and capacity rarely occur alone, *i. e.* independently of one another, but mostly interdependent, *viz.* the change of one of them involves changes of one or all others. The need of an acoustic standard became a claim after SCHUSTER's construction of an acoustic Wheatstone bridge[1]. Also at Breslau, in WAETZMANN's laboratory, a variable acoustic impedance[2] seems to have been constructed which, however, was troublesome to handle and which, moreover, was only approximately correct; its description never was published. The impedance used by SCHUSTER in his bridge is continuously variable; however, its calculation is rather complicated[3] and it is difficult to make it comprise both the small absorption coefficients from 0 to 10 %; and the great ones exceeding 90 %; finally, its reactive part (the felt tube) and its ohmic part (the piston tube) do not work quite independently of one another. A variable, radiation-damped acoustic impedance including the impedance values which probably exist for the human ear was earlier suggested by THORSEN[4].

In the following, the writer wishes to account in detail for the construction of a variable acoustic impedance which is relatively simple both in its mode of action and its manipulation and

1) K. SCHUSTER: Phys. Zt. **35**, 408, 1934.
2) K. SCHUSTER: E. N. T. **13**, 164, 1936.
3) K. SCHUSTER and W. STÖHR: Akust. Ztschr. **4**, 253, 1939.
4) V. THORSEN: D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. XX, 9, 1943, (in the following denoted as Essay I).

which, furthermore, covers a large range of absorption, *viz.* that from practically 100 % nearly down to 0 %. Besides, it is almost purely radiation-damped, whence its damping is easily measurable, for example, with a condenser microphone. This is of importance not only when its calculations are to be controlled, but it will likewise increase its range of applicability.

1. The Tube as Acoustic Impedance.

According to the theory of the acoustic tube-line, every smooth tube without dissipation represents an acoustic impedance of the magnitude

$$\frac{Z_i}{\varrho c} = \frac{\frac{Z_u}{\varrho c}\cos kl + i \sin kl}{\cos kl + i \frac{Z_u}{\varrho c}\sin kl} \text{ [1]}. \tag{1}$$

Here, $\frac{Z_i}{\varrho c}$ is the inlet impedance, *i. e.* the impedance at the mouth of a tube of length l, which at the other end is terminated by the impedance $\frac{Z_u}{\varrho c}$ (the outlet impedance). When introducing the amplitude reflection coefficient r of the outlet impedance, which is determined by

$$re^{i\vartheta} = \frac{\frac{Z_u}{\varrho c} - 1}{\frac{Z_u}{\varrho c} + 1},$$

and its phase change ϑ, (1) may also be written

$$\frac{Z_i}{\varrho c} = \frac{1 - r^2 - i\, 2\, r \sin(2\, kl - \vartheta)}{1 + r^2 - 2\, r \cos(2\, kl - \vartheta)}. \tag{2}$$

[1] Cf. V. Thorsen: D. Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. XX, 10, 1943 (in the following denoted as Essay II).

$\frac{Z_u}{\varrho c}$ then being written in the form $w_0 + iq_0$, (1) passes into

$$\frac{Z_i}{\varrho c} = \frac{w_0 + i\left(q_0 \cos 2\,kl - \frac{1}{2}\,[w_0^2 + q_0^2 - 1]\sin 2\,kl\right)}{(\cos kl - q_0 \sin kl)^2 + w_0^2 \sin^2 kl}. \tag{3}$$

Thus, we have

$$\frac{Z_i}{\varrho c} = w + iq\,,$$

where

$$\left.\begin{aligned} w &= \frac{w_0}{(\cos kl - q_0 \sin kl)^2 + w_0^2 \sin^2 kl}\,, \\ q &= \frac{q_0 \cos 2\,kl - \frac{1}{2}\,[w_0^2 + q_0^2 - 1]\sin 2\,kl}{(\cos kl - q_0 \sin kl)^2 + w_0^2 \sin^2 kl}. \end{aligned}\right\} \tag{4}$$

Finally, when introducing the energy absorption coefficient a, we find

$$a = \frac{4\,w}{(w+1)^2 + q^2} \quad \text{and} \quad \operatorname{tg}\vartheta = \frac{2\,q}{w^2 + q^2 - 1}.\text{[1)]} \tag{5}$$

The equations (5) in a (w, q) coordinate system with a and $\operatorname{tg}\vartheta$ as parameters represent the known system of circles intersecting one another at right angles, as shown in Fig. 1.

We shall now look at some simple cases. For the absolutely rigid, perfectly reflecting wall, we have

$$a = 0,$$

i. e. the very q axis and, therefore, $w = 0$. Further, $\vartheta = 0$, *i. e.* an infinite large phase circle, whence $q = \infty$. For the absolutely compliant wall, we again have

$$a = 0,$$

1) Cf. Essay I, p. 5, and Essay II, p. 9.

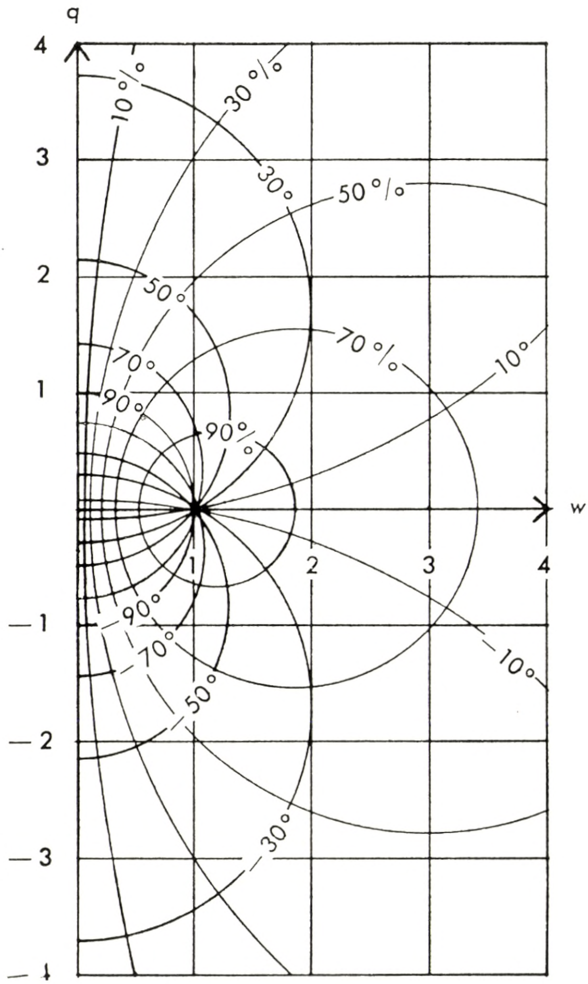

Fig. 1.

hence, also $w = 0$, but in this case $\vartheta = \pm \pi$, *i. e.* $q = 0$. The point (1.0) corresponds to a substance of the same acoustic resistance as the air.

Summarizing, we thus have,

for the rigid wall: $w = 0, \quad q = \infty,$

for the compliant wall: $w = 0, \quad q = 0.$

For the perfectly absorbing substance: $w = 1, \quad q = 0.$

Now, in a tube, we have a combination of acoustic elements, generally both phase change and absorption. If the tube is closed at one end (Fig. 2) with a rigid 100 % reflecting plate,

$$\frac{Z_u}{\varrho c} = w + iq,$$

where

$$w = 0, \quad q = \infty.$$

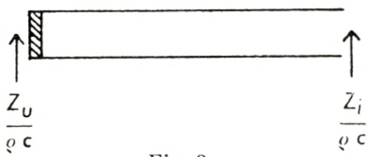

Fig. 2.

The inlet impedance Z_i then, according to (1), becomes

$$\frac{Z_i}{\varrho c} = \frac{\cos kl + i \dfrac{1}{\dfrac{Z_u}{\varrho c}} \sin kl}{\dfrac{1}{\dfrac{Z_u}{\varrho c}} \cos kl + i \sin kl} = -i \cot kl. \tag{6}$$

If $l = 0$, we find $\frac{Z_i}{\varrho c} = \infty$, which result is obvious. If $l = \frac{\lambda}{4}$, $Z_i = 0$, a result which also should be evident, since the tube now is a closed organ pipe measuring $^1/_4$ wave-length. The tube thus represents a pure reactive acoustic resistance which, if l varies from 0 to $\frac{\lambda}{4}$, itself varies from 0 to ∞.

In the same way, the open tube (Fig. 3) can be treated as an acoustic resistance. If we primarily suppose that the mouth represents an entirely compliant wall, we find for $\frac{Z_u}{\varrho c}$

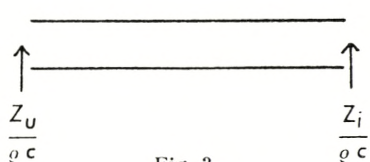

Fig. 3.

$\frac{Z_u}{\varrho c} = w + iq$, where $w = 0$, $q = 0$, which, inserted in (1), gives

$$\frac{Z_i}{\varrho c} = \frac{i \sin kl}{\cos kl} = i \operatorname{tg} kl. \tag{7}$$

For $l = 0$ and $l = \frac{\lambda}{2}$ become $\frac{Z_i}{\varrho c} = 0$, which is also quite obvious. In the latter case, the tube is an open organ pipe measuring $^1/_2$ wave-length. For $l = \frac{\lambda}{4}$, $\frac{Z_i}{\varrho c} = \infty$, whence this acoustic impedance too is purely reactive, varying between 0 and ∞.

In electric analogy, the tube thus in an acoustical conduit acts as a pure reactance (Fig. 4), and the electric generator from the field of electricity may thus be an acoustic generator, for example a telephone or another sound source sending its sound

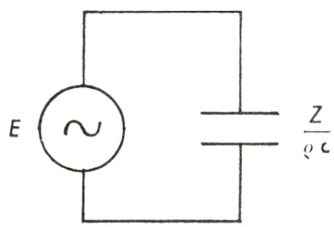

Fig. 4.

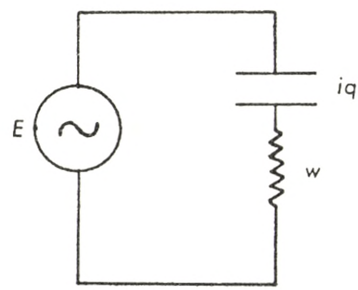

Fig. 5.

energy into the tube. In reality, however, the mouth of an open tube will not be a perfectly compliant wall, so that we have a w of a given, though small, value; q, consequently, is not either quite equal to zero. Thus, neither the real nor the imaginary part of $\frac{Z_i}{\varrho c}$ becomes equal to zero, if $l = \frac{\lambda}{2}$ and the electric comparison picture appears as in Fig. 5. If the tube length varies, w changes along an iso-absorption circle in Fig. 1, and from (4) it is seen that the impedance becomes real, if

$$q_0 \cos 2\,kl - \frac{1}{2}\left[w_0^2 + q_0^2 - 1\right] \sin 2\,kl = 0,$$

or

$$\operatorname{tg} 2\,kl = \frac{2\,q_0}{w_0^2 + q_0^2 - 1} \quad \text{(cf. the 2nd equation (5)).}$$

The corresponding values of w (w_1 and w_2), if $q = 0$, thus become the two values where the iso-absorption circle intersects the w axis. For these values it holds that $w_1 w_2 = 1$. Thus, the tube as an acoustic impedance is likewise arranged so that, with varying lengths, both the real and the imaginary parts change, however, in such a way that the tube has the same absorption coefficient for all values of l. Hence, w and q change simultaneously so that

$$a = \frac{4\,w}{(w+1)^2 + q^2} = \text{const.}$$

Therefore it may be used as absorption standard, although the values of w are different for each tube length.

A presupposition for the correctness of the above statement is that no other ohmic resistances occur than the radiation resistance. If the tube is so narrow or so long that further dissipation resistance is found in the form of friction, the relations become different. This possibility is treated in detail elsewhere[1].

2. Other Combinations of Tube Impedances.

If we have a combined system of tubes, as shown in Fig. 6, it must obviously have certain peculiar properties. It consists chiefly of a tube which, for theoretical reasons, we imagine to be divided into two parts, l_1 and l_3, and in whose joining-plane (A) is placed a side-branch which, with the help of a tight-fitting metal piston, may be given different lengths (l_2). A sound-wave which enters from the right will divide into two parts at A. That part which enters the side-branch is entirely reflected from the piston and returns to A with a phase difference relative to that passing on through l_3. If l_2 is exactly equal to $\frac{\lambda}{4}$, the wave reflected from the piston will, when reaching A, be exactly in the phase opposite to that which passes

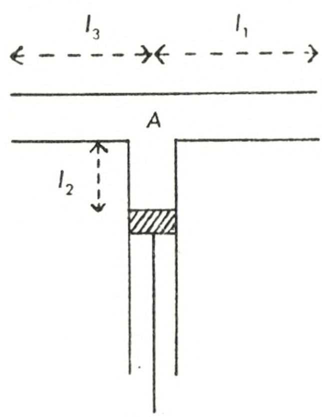

Fig. 6.

1) Essay I, p. 8 *et seq.*, and Essay II, p. 13 *et seq.*

along l_3, and will thus completely extinguish this wave. In this case, the impedance of the side-branch will short-circuit the impedance of l_3. If the piston is pushed up to A, no effect of the side-branch is observed. Its impedance is infinitely great and connected in parallel to the impedance of l_3. Other peculiar relations should be found if either $l_2 + l_3$ or $l_1 + l_3$ are multiples of half wave-lengths. This side-branch principle was set up by QUINCKE[1)].

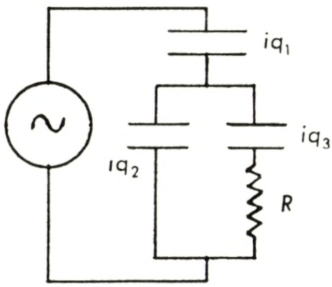

Fig. 7.

The electric equivalent is shown in Fig. 7. R is the radiation at the open (left) end of l_3. If q_2 is equal to ∞, q_1 and q_3 are connected in series; if $q_2 = 0$, R and q_3 are short-circuited and the impedance is equal to iq_1. Roughly spoken, this manner of connecting seems thus to satisfy the above stated demands; yet there are some difficulties which will appear from the following calculation of the impedance.

We find

$$\left.\begin{aligned} Z &= iq_1 + \frac{iq_2\,(R + iq_3)}{iq_2 + R + iq_3} \\ &= \frac{Rq_2^2}{R^2 + (q_2 + q_3)^2} + i\,\frac{R^2\,(q_1 + q_2) + (q_2 + q_3)\,(\Sigma q)}{R^2 + (q_2 + q_3)^2}, \end{aligned}\right\} \quad (8)$$

putting as a simplification

$$q_1 q_2 + q_1 q_3 + q_2 q_3 = (\Sigma q).$$

Further, we find

$$|Z|^2 = \frac{R^2\,(q_1 + q_2)^2 + (\Sigma q)^2}{R^2 + (q_2 + q_3)^2}.$$

From (8) it is evident that Z is not far from being real, if

$$q_2 + q_3 = 0.$$

Therefore it is worth while examining this case somewhat more closely. We find

1) G. H. QUINCKE: Pogg. Ann. **128**, 177, 1866.

$$Z = \frac{q_2^2}{R} + i(q_1 + q_2). \tag{9}$$

This simplified impedance according to (5) has the absorption coefficient

$$a = \frac{4\,\dfrac{q_2^2}{R}}{\left(\dfrac{q_2^2}{R} + 1\right)^2 + (q_1 + q_2)^2} = \frac{4\,Rq_2^2}{(R + q_2^2)^2 + R^2\,(q_1 + q_2)^2}.$$

If R is small, and particularly if $q_1 + q_2$ moreover is small, this latter expression becomes

$$a = \frac{4\,Rq_2^2}{(R + q_2^2)^2}. \tag{10}$$

Under these presuppositions, *i. e.* $q_2 + q_3 = 0$ and R small, (9) represents a pure ohmic acoustic resistance with an absorption coefficient given in (10). Forming $\dfrac{da}{dq_2}$, we find

$$\frac{da}{dq_2} = 4\,R\,\frac{(R + q_2^2)^2 \cdot 2\,q_2 - q_2^2\,(R + q_2^2) \cdot 2\,q_2}{(R + q_2^2)^4}, \tag{11}$$

which, put equal to zero, besides $q_2 = 0$ gives

$$R = q_2^2. \tag{12}$$

If this condition is fulfilled, we find

$$a = 1,$$

i. e. 100 % absorption. Now, if q_2 varies so that $q_2 + q_3$ still is equal to zero (q_3 must thus likewise be changed), a varies in accordance with (10). Then it depends to some degree on the rate at which a varies with q_2 for according to (9), Z is no longer purely ohmic if $q_1 + q_2$ deviates considerably from 0. This question, however, will be easiest to elucidate on the basis of experiments, and experiments show that Z within fairly wide absorption limits may be regarded as purely ohmic. This is supported if Z in its tube lengths is arranged so that, at 100 % absorption where $l_2 + l_3 = \dfrac{\lambda}{4}$, $l_1 + l_2$ is somewhat greater than $\dfrac{\lambda}{4}$.

When q_2 increases, *i. e.* l_2 decreases $\left(\text{however, } l_2 + l_3 = \frac{\lambda}{4}\right)$, $q_1 + q_2$ will pass from a small positive value through 0 to a small negative value. In this way the range of absorption of the impedance is increased essentially. How far we may go can be decided with the aid of experiments.

We can also see theoretically that the imaginary part of Z is of minor importance. Since, from (9), we find

$$\operatorname{tg} \vartheta = \frac{(q_1 + q_2) R}{q_2^2}, \tag{13}$$

and, if $\frac{d(\operatorname{tg}\vartheta)}{dq_2}$ is formed, we find

$$\frac{d(\operatorname{tg}\vartheta)}{dq_2} = R \frac{q_2^2 \cdot 1 - (q_1 + q_2) \cdot 2 q_2}{q_2^4} = -R q_2 \frac{2 q_1 + q_2}{q_2^4},$$

which, besides $q_2 = 0$, gives

$$q_2 = -2 q_1. \tag{14}$$

If q_1 is not chosen too small, this value for q_2 brings (12) to a fairly flat minimum of a shape represented in Fig. 8. Experiments also

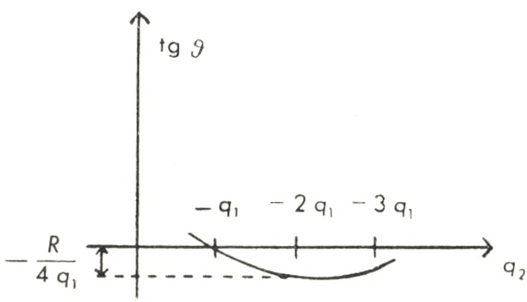

Fig. 8.

prove this to be correct. Evidently, the value (12) gives a maximum for (10)—which should indeed be obvious—since, if q_2 increases from a value smaller than R through R and to a value greater than R, $\frac{da}{dq_2}$ passes from plus through 0 to minus. It appears from (9) that, with decreasing q_2, the ohmic resistance $\frac{q_2^2}{R}$, from being greater than 1, becomes equal to 1 (*viz.*, for $R = q_2^2$) and finally assumes values below 1. Since two values of $\frac{q_2^2}{R}$, the product of which is equal to 1, lead to the same absorption coefficient, the absorption coefficient as a function of l_2 must become a somewhat symmetrical curve with a maximum for $q_2^2 = R$. It must commence in the vicinity of zero, if l_2 is very small, and again approximate zero, if l_2 approximates $\frac{\lambda}{4}$. It is also important to note that it is possible to get all resistance values, both those greater than 1 and those smaller than 1; however, the greater an absorption coefficient we want to obtain at the maximum, the shorter is the range over which q_2 varies, if resistance values smaller than 1 are wanted, since R is a small magnitude, and the resistance is equal to $\frac{q_2^2}{R}$. The falling branch of the absorption coefficient curve as a function of l_2 therefore becomes very steep. If it is unnecessary to obtain as much as 100 $^0/_0$ absorption, the falling branch may be made less steep, the adjustment thus becoming less critical. These relations are also substantiated by the measurements.

The impedance determined by (9) we may call the central point impedance of the combined tube-system.

If we look for inlet and outlet impedances in the two tubes l_1 and l_3, respectively, the former is identical with the inlet impedance in an open tube of length l_3. The inlet impedance is thus determined by the tube which is situated to the left of the central point impedance, the outlet impedance by the tube which is situated to the right of the central point impedance (Fig. 9). The impedance at a given value of the absorption coefficient varies as a function of the tube lengths l_1 and l_3 along the absorption circle determined by $\frac{q_2^2}{R}$, and the w and q values of the inlet impedance are found by intersection of this absorption circle with the line $q = \text{tg}\, kl_3$; in the same way the w and q values of the outlet impedance are found as the intersection of the absorption circle with the line $q = \text{tg}\, kl_1$. The q values of the inlet and the outlet impedances naturally become q_3 and q_1. The shorter the tube lengths l_1 and l_3, the more the two impedances approximate one another and, at the same time, the central point impedance. If $l_1 = l_3$, the inlet and the outlet impedances are equal. However, not before both are equal to zero does the impedance become reactance-free and thus purely ohmic. This is not quite realizable in practice, but if the pipes l_1 and l_3 are

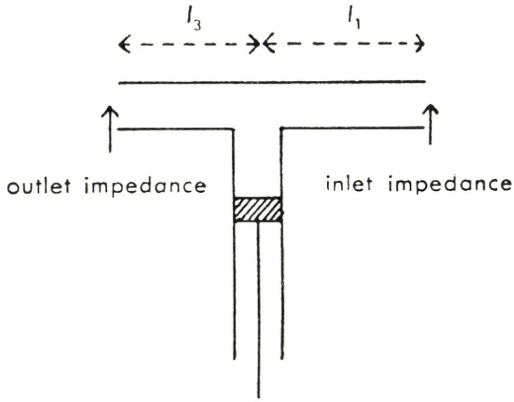

Fig. 9.

short[1], it can approximately be realized, the approximation being best at low frequencies.

We can also get an idea of the highest absorption coefficient to be obtained with given tube lengths. Suppose, for example, that the longest side-branch has the length l, *i. e.* the q value

$$q_A = \operatorname{tg} kl.$$

If this straight line, which runs parallel with the w axis, is brought to intersect the absorption circle system whose equation with the absorption coefficient a as parameter may be written

$$w^2 + q^2 - 2\,w\,\frac{2-a}{a} + 1 = 0,$$

we find

$$w = \frac{2-a}{a} \pm \sqrt{\left(\frac{2-a}{a}\right)^2 - 1 - q_A^2}\,.$$

Real intersection thus is conditioned by

$$\left(\frac{2-a}{a}\right)^2 - 1 + q_A^2 \geq 0$$

1) Here as well as elsewhere 'short tube lengths' means short in proportion to the wave-lengths or, in other words, kl is a small angle.

or by

$$a \leqq \frac{2}{1 + \sqrt{1 + q_A^2}}.$$

Accordingly, the sign of equation leads to the highest a-value which the impedance can attain at a given tube length and which thus corresponds to the line $q_A = \operatorname{tg} kl$ just touching the absorption circle.

3. The Influence of Dissipation Damping.

Since it is impossible to keep all the tube lengths short, it is necessary to examine what damping due to friction in the tubes means. This examination claims a picture of comparison

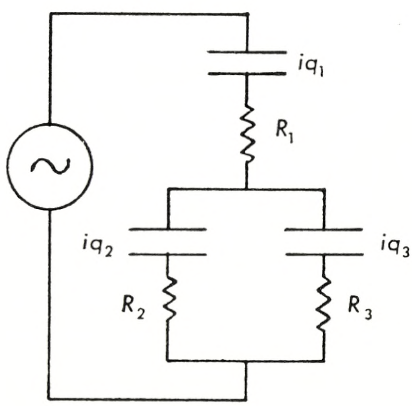

Fig. 10.

as shown in Fig. 10. For the impedance of this system we find

$$Z = R_1 + iq_1 + \frac{(R_2 + iq_2)(R_3 + iq_3)}{R_2 + R_3 + i(q_2 + q_3)}$$

$$= \frac{R_2 R_3 - (\Sigma q) + R_1 (R_2 + R_3) + i\left[(R_2 + R_3) q_1 + R_3 q_2 + R_2 q_3 + R_1 (q_2 + q_3)\right]}{R_2 + R_3 + i(q_2 + q_3)}$$

and, after multiplication of the numerator and the denominator with

$$R_2 + R_3 - i(q_2 + q_3)$$

and some reduction

$$Z = w + iq,$$

where

$$\left.\begin{aligned} w &= \frac{R_2 R_3 (R_2 + R_3) + R_2 q_3^2 + R_3 q_2^2 + R_1 (R_2 + R_3)^2}{(R_2 + R_3)^2 + (q_2 + q_3)^2} \\ q &= \frac{(R_2 + R_3)^2 q_1 + R_2^2 q_3 + R_3^2 q_2 + (q_2 + q_3)(\Sigma q) - R_1 (R_2 + R_3)(q_2 + q_3)}{(R_2 + R_3)^2 + (q_2 + q_3)^2}. \end{aligned}\right\} \tag{15}$$

In case $q_2 + q_3 = 0$, Z assumes the form

$$Z = \frac{R_2 R_3 + q_2^2}{R_2 + R_3} + R_1 + i\left(q_1 + \frac{R_3 - R_2}{R_2 + R_3} q_2\right) \tag{16}$$

and the absorption coefficient in the first approximation becomes

$$a = \frac{4\left[(R_2 R_3 + q_2^2)(R_2 + R_3) + R_1 (R_2 + R_3)^2\right]}{(R_2 R_3 + q_2^2 + R_1 (R_2 + R_3) + R_2 + R_3)^2}. \tag{17}$$

Just as in the simpler case, where R_1 and R_2 were assumed to be equal to zero, we shall find the maximum for a, when a is regarded as a function of q_2. Thereby we find

$$\frac{da}{dq_2} = 4 \cdot \frac{(R_2 R_3 + q_2^2 + R_1 (R_2 + R_3) + R_2 + R_3)\, 2 q_2}{N^2}$$
$$- 4 \cdot \frac{[R_2 R_3 + q_2^2 + R_1 (R_2 + R_3)] \cdot 2 [R_2 R_3 + q_2^2 + R_1 (R_2 + R_3) + R_2 + R_3] \cdot 2 q_2 (R_2 + R}{N^2}$$

which, put equal to zero, besides $q_2 = 0$ gives

$$q_2^2 = R_2 + R_3 - R_1 R_2 - R_2 R_3 - R_1 R_3 = R_2 + R_3 - (\Sigma R); \tag{18}$$

for the sake of simplicity, we put

$$R_1 R_2 + R_2 R_3 + R_1 R_3 = (\Sigma R),$$

i. e. an expression corresponding to (12). If (18) is inserted in (17), we also find $a = 1$, just as before.

R_2 and R_3 being small, (ΣR) is a small magnitude in proportion to R_2 and R_3, whence (17) in good approximation reaches a maximum for

$$q_2^2 = R_2 + R_3.$$

In other words this means that, in (16), R_2R_3 and R_1 may be disregarded so that (16) assumes the following form:

$$Z = \frac{q_2^2}{R_2 + R_3} + i\left(q_1 + \frac{R_3 - R_2}{R_2 + R_3}\, q_2\right). \tag{19}$$

4. Experimental Results.

In order to test the correctness of the formulae developed in the preceding sections, the writer has performed a series of experiments with an impedance of the form shown in Fig. 6. The impedance was made from brass tubes with a lumen measuring 6 mm. in diameter. The tube length l_1 was 2.9 cm., the shortest tube length l_3 being 1.1 cm. The latter could be lengthened with additional tubes of known lengths. The length of l_2 was varied with a piston which, in order to ensure tightness, was supplied with a piston ring consisting of a piano string. The impedance was connected to a calibrated Schuster bridge. Absorption and phase of the inlet impedance were measured for altogether 14 different lengths of l_3, absorption and phase for each value of l_3 were determined as a function of l_2. The frequency applied was 768 Hz, $\lambda = 44.3$ cm. For the determination of every single absorption and phase curve, measurements were performed for about 30 different values of l_2, particularly close around the maximum. Fig. 11 represents an example of the results obtained in a series of measurements of an absorption curve, and Fig. 12 shows a similar measurement for the corresponding phase curve. The accuracy is extraordinarily satisfactory, c. $^1/_2$ $^0/_0$ for the absorption coefficients, and a few degrees for the corresponding phases.

Total results for the maximum absorption coefficients and the corresponding phases are recorded in Table 1; the head lines of the columns are supplied with easily comprehensible symbols referring to those used above. The calculations of a and ϑ are based upon the formulae (5). Finally, the calculations are illustrated by Figs. 13 and 14.

Fig. 13 is very instructive, showing that, for $l_3 = 1.4$ cm., $l_2 = 9.35$ cm., with approximation $R_2 = q_2^2$ so that we here have

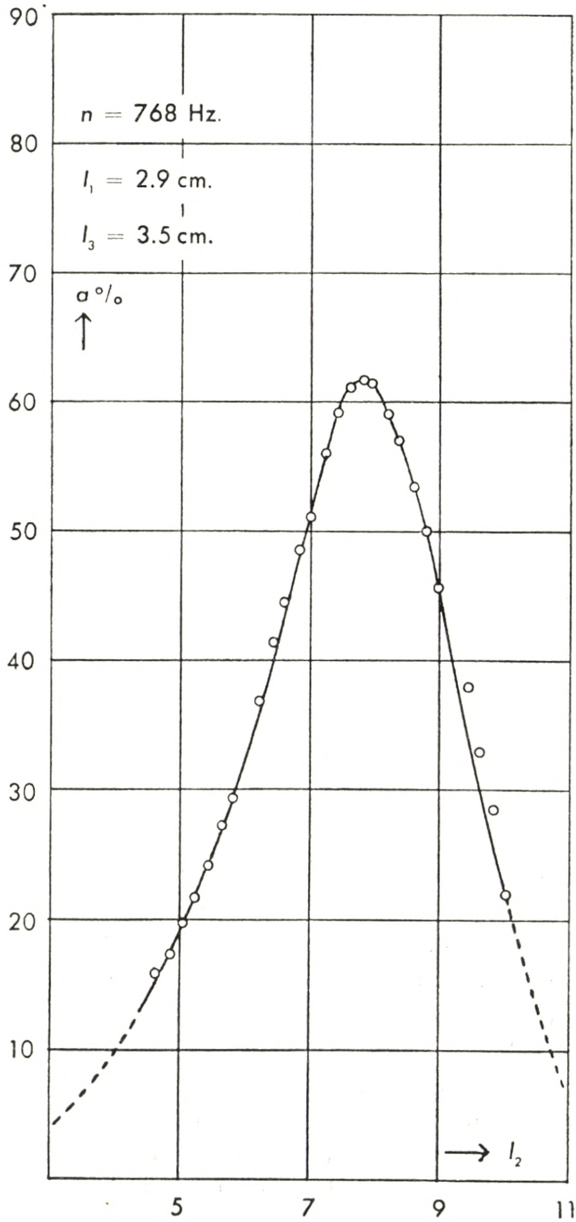

Fig. 11.

Table 1.

l_3	l_2	q_3	q_2	$\frac{q_2^2}{R}$	$\frac{R}{q_2^2}$	a % Calculated	a % Observed	ϑ^0 Calculated	ϑ^0 Observed
5.5	6.8	1.00	0.695	7.91	0.127	39.5	40.5	ca.–90	–90
4.5	7.4	0.740	0.575	5.41	0.184	52.0	52.7	–74.3	–80
3.5	7.8	0,542	0.505	4.16	0.240	61.7	61.5	–59.1	–60
2.9	8.15	0.440	0.445	3.22	0.310	72.0	73.0	–51.0	–53
2.5	8.5	0.375	0.384	2.41	0.415	82.0	83.5	–47.5	–53
2.0	8.85	0.300	0.327	1.74	0.575	92.5	94.5	–41.2	–40
1.8	9.0	0.265	0.300	1.47	0.680	96.0	96.0	–44.0	–35
1.7	9.1	0.250	0.290	1.36	0.736	97.4	97.5	–50.0	–55
1.6	9.2	0.235	0.275	1.23	0.815	98.5	98.0	–58.0	–60
1.5	9.3	0.220	0.260	1.10	0.91	98.5	98.5	–76.6	—
1.4	9.35	0.200	0.250	1.03	0.97	99.5	99.0	ca.–90	—
1.3	9.45	0.190	0.235	0.90	1.11	98.8	99.0	+	+
1.2	9.5	0.175	0.230	0.85	1.18	98.0	97.5	+	+
1.1	9.65	0.160	0.205	0.69	1.45	95.5	97.0	+	+

Hence, a value must be found for l_2, where $q_3^2 + \left(\frac{R}{q_2^2}\right)^2 = 1$, and the denominator therefore is equal to 0, tg $\vartheta = -\infty$, $\vartheta = -90°$. This point was fortuitously found for $l_3 = 5.5$ cm., $l_2 = 6.8$ cm. If l_2 increases, q_3^2 decreases, while $\frac{R}{q_2^2}$ increases slowly. Here we have

$$q_3^2 + \left(\frac{R}{q_2^2}\right)^2 < 1,$$

whence ϑ becomes negative. For a given value of l_2, we now find on account of the increase in $\frac{R}{q_2^2}$

$$q_3^2 + \left(\frac{R}{q_2^2}\right)^2 = 1,$$

and ϑ again becomes $-90°$. This holds for $l_3 = 1.5$ cm., $l_2 = 9.3$ cm. Hence, in the interval ϑ must have had a (numerical) minimum, *viz.*, for $l_3 = 2.0$ cm., $l_2 = 8.8$ cm. At still higher values for l_2, tg ϑ becomes positive, *viz.*, if

$$q_3^2 + \left(\frac{R}{q_2^2}\right)^2 > 1.$$

This is indicated by the dotted branch of the phase curve in the lower part of Fig. 14. The curve presents, however, a very sharp bend, and the accuracy is but small. Actually the experiments only show that the phase becomes positive. Thus a discontinuity in the phase curve occurs. A closer examination of this relation, which could only be found within the very greatest absorption range between 99 to 100 $^0/_0$, is in progress.

It is clear from the preceding account that it will be possible to produce an impedance which can be brought to assume all possible values of absorption and phase. It is a characteristic feature of this impedance that its damping is a pure radiation damping. This means that we might be able to measure the exact effect, emitted from a telephone, on the human ear. If this impedance is known, and measured for example by means of a Schuster bridge, the variable impedance is adjusted to the value and placed before the telephone. Then the radiation of this telephone through the impedance is equal to the effect produced on the ear. A solution of this problem was suggested before[1], and the program of an investigation was briefly as follows.

(1) The radiation curve of a telephone is measured for a series of frequencies.
(2) The impedance of the ear for the same frequencies is measured with a Schuster bridge.
(3) The variable impedance is adjusted for each frequency as equal to the impedance of the ear.
(4) The impedance is placed before the telephone and, subsequently, the radiation is re-measured.

If the distance between telephone and measuring apparatus (condenser microphone) in case 1 and case 4 is the same, information is obtained as to how great a fraction of the effect, which the telephone is able to emit, is absorbed by the ear. Such an impedance will be much easier to handle than the previously suggested one, and it will moreover be possible to make it cover a far greater range. It must, of course, be adjusted to different standard frequencies, and therefore we possibly may be compelled to make a compromise between the lowest frequency to

[1] Essay I, p. 18.

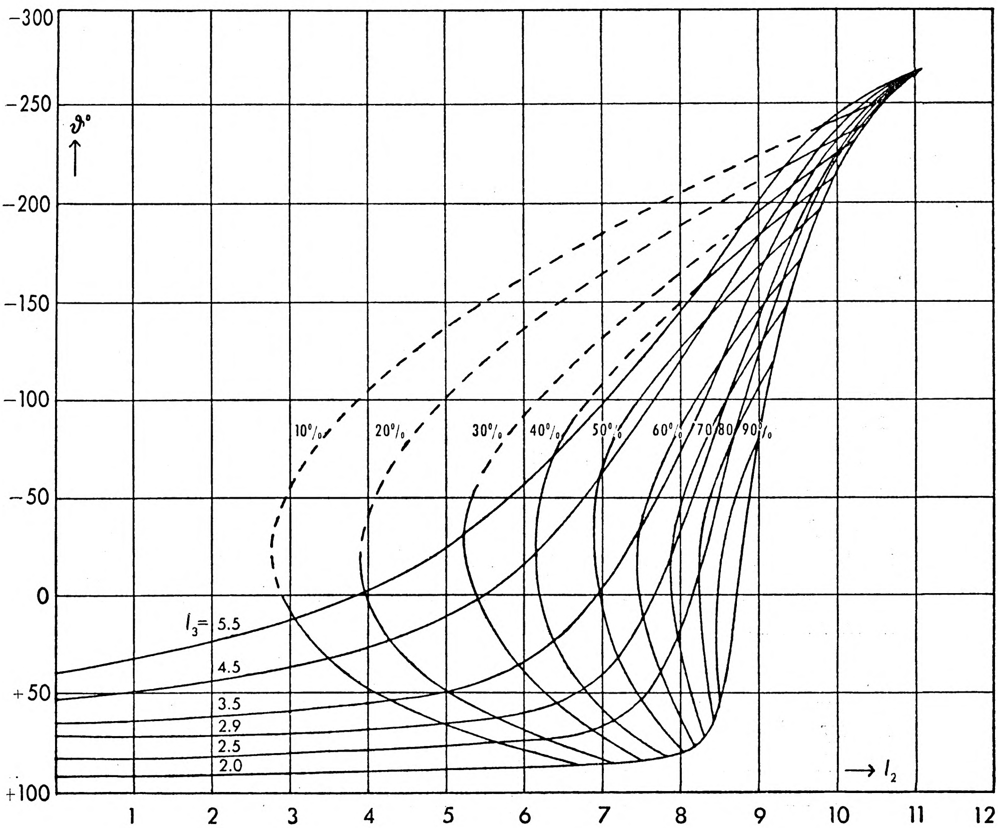

Fig. 15.

be used and the greatest geometric range of the impedance to be allowed.

Fig. 15 was plotted on the basis of all measurements performed. On a series of curves representing the phase as a function of l_2 for different values of l_3 iso-absorption curves for 10, 20, 30, ... 90 $^0/_0$ of absorption are inserted. From these multitudes of curves we may infer, possibly by interpolation, which values l_2 and l_3 must obtain in order to yield a given absorption coefficient and a given phase angle. If we want, for example, $a = 25\ ^0/_0$, $\vartheta = 60°$, we evidently need (about) $l_2 = 5.2$ cm. and $l_3 = 3.3$ cm. In the impedance to be constructed both l_2 and l_3 must be continuously variable. Such an impedance has already been produced, and it has proved to comply with our expectations.

As soon as some still unexplained, however less important details are elucidated, an examination of patients is planned.

Summary.

With the aid of a system of acoustic tube impedances a variable acoustic impedance which covers a rather large range of absorption coefficients and phase changes could be constructed and partly calculated. In order to support the theory, numerous measurements of the values of the inlet impedance of the variable impedances were performed. Particularly good agreement with the theoretical expectance was obtained and it also appears from the measurements that the accuracy is significant. The calculations were performed for tubes both with and without dissipation. It is intended to apply the impedance, *inter alia*, to an objective determination of the effect emitted from a telephone on the human ear.

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